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On the Forcing Domination and the Forcing Total Domination Numbers of a Graph

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Abstract
Let G be a connected graph with at least two vertices and S a γ_t-set of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique γ_t-set containing T. The *forcing total domination number* of S, denoted by $f_{\gamma_t}(S)$, is the cardinality of a minimum forcing subset of S. The *forcing total domination number* of G, denoted by $f_{\gamma_t}(G)$ is defined by $f_{\gamma_t}(G) = \min\{f_{\gamma_t}(S)\}$, where the minimum is taken over all minimum total dominating sets S in G. Some general properties satisfied by this concepts are studied. The forcing total dominating number of certain standard graphs are determined. It is shown that for every pair a, b of integers with $0 \leq a < b$ and $b \geq 1$, there exists a connected graph G such that $f_{\gamma_t}(G) = a$ and $\gamma_t(G) = b$, where $\gamma_t(G)$ is total domination number of G. It is also shown that for every pair a,b of integers with $a \geq 0$ and $b \geq 0$, there exists a connected graph G such that $f_{\gamma_t}(G) = a$ and $f_\gamma(G) = b$, where $f_\gamma(G)$ is the forcing domination number of G.

Keywords Domination number · Total domination number · Forcing domination number · Forcing total domination number

Mathematics Subject Classification 05C69

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1 Introduction

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic definitions and terminologies we refer to [4]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. The *open neighbourhood of a vertex v* in a graph G is defined as the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, while the *closed neighbourhood* of v in G is defined as $N_G[v] = N_G(v) \cup \{v\}$. For any vertex v in a graph G, the number of vertices adjacent to v is called the *degree* of v in G, denoted by $deg_G(v)$. If the degree of a vertex is 0, then it is called an *isolated vertex*, while if the degree is 1, it is called an *end-vertex*. The *minimum degree* of vertices in G is defined by $\delta(G) = min\{deg(v) : v \in V(G)\}$. The *maximum degree* of vertices in G is defined by $\triangle(G) = max\{deg(v) : v \in V(G)\}$. A vertex v is called a *universal vertex* if $deg_G(v) = n - 1$. For any set S of vertices of G, the *induced subgraph* $G[S]$ is the maximal subgraph of G with vertex set S.

A subset $S \subseteq V(G)$ is called a *dominating set* if every vertex $v \in V(G) \backslash S$ is adjacent to a vertex $u \in S$. The *domination number* $\gamma(G)$ of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called a γ-*set* of G.

A vertex v of a connected graph G is said to be a *dominating vertex* of G if v belongs to every γ-set of G. If G has a unique γ-set S, then every vertex of S is a dominating vertex of G. A *total dominating set* of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of total dominating sets S in G. A minimum total dominating set of a graph G is hence often called a γ_t-*set* of G. These concepts were studied in [2, 3, 5–7].

The forcing set in a graph is a very interesting concept. In the management of a company, the executive committee consists of senior members who have adequate rapport with other members of the company. Some members of the executive committee may sit in other important committees also. Sometimes, restrictions are imposed on members that they can be part of exactly one committee. This precisely leads to the concept of the forcing set. Let S be a γ-set of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique γ-set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing domination number* of S, denoted by $f_\gamma(S)$, is the cardinality of a minimum forcing subset of S. The *forcing domination number* of G, denoted by $f_\gamma(G)$, is $f_\gamma(G) = min\{f_\gamma(S)\}$, where the minimum is taken over all γ-sets S in G. The forcing concept was first introduced and studied in minimum dominating sets in [1]. Many authors have studied this forcing concept with respect to several parameters like domination, geodetic, Steiner, hull, detour, monophonic, etc. In this paper we study the forcing concept with respect total domination. Throughout the following G denotes a connected graph with at least two vertices. The following theorem is used in the sequel.

Theorem 1 [1] *Let G be a connected graph and W be the set of all dominating vertices of G. Then* $f_\gamma(G) \leq \gamma(G) - |W|$

2 The Forcing Total Domination Number of a Graph

Definition 1 Let G be a connected graph with at least two vertices and S a γ_t-set of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique γ_t-set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing total domination number* of S, denoted by $f_{\gamma_t}(S)$, is the cardinality of a minimum forcing subset of S. The *forcing total domination number* of G, denoted by $f_{\gamma_t}(G)$ is defined by $f_{\gamma_t}(G)$ = min $\{f_{\gamma_t}(S)\}$, where the minimum is taken over all minimum total dominating sets S in G.

Note 1 A forcing set T of vertices of G uniquely determines a γ_t-set containing T.

Example 1 For the graph G given in Fig. 1, $S_1 = \{v_2, v_3\}$ is the unique γ_t-set of G so that $\gamma_t(G) = 2$ and $f_{\gamma_t}(S_1) = 0$. Also $S_1 = \{v_2, v_3\}$, $S_2 = \{v_1, v_3\}$ and $S_3 = \{v_2, v_4\}$ are the only three γ-sets of G such that $f_\gamma(S_1) = 2$, $f_\gamma(S_2) = f_\gamma(S_3) = 1$ so that $\gamma(G) = 2$ and $f_\gamma(G) = 1$. For the graph G given in Fig. 2, $M_1 = \{v_2, v_3, v_4\}$, $M_2 = \{v_1, v_2, v_3\}$, $M_3 = \{v_1, v_4, v_6\}$ and $M_4 = \{v_4, v_5, v_6\}$ are the only four γ_t-sets of G such that $f_{\gamma_t}(M_1) = f_{\gamma_t}(M_2) = f_{\gamma_t}(M_3) = 2$ and $f_{\gamma_t}(M_4) = 1$ so that $\gamma_t(G) = 3$ and $f_{\gamma_t}(G) = 1$. Also $M_5 = \{v_2, v_4\}$ is the unique γ-set of G so that $\gamma(G) = 2$ and $f_\gamma(M_5) = 0$.

The next theorem follows immediately from the definition of the total domination number and the forcing total domination number of a connected graph G.

Theorem 2 *For any connected graph* G, $0 \le f_{\gamma_t}(G) \le \gamma_t(G)$.

In the following we determine the forcing total domination number of some standard graphs.

Theorem 3 *For the path* $G = P_n (n \ge 4)$,

$$f_{\gamma_t}(P_n) = \begin{cases} 0 & \text{if } n = 5 \text{ or } n \equiv 0 \pmod 4 \\ 1 & \text{if } n \text{ is odd and } n \ne 5 \\ 2 & n \equiv 2 \pmod 4 \end{cases}.$$

Proof Let $V(P_n) = \{v_1, v_2, ..., v_n\}$.

Case 1. n is odd.

Subcase 1 (i). Let $n = 5$. Then $S = \{v_2, v_3, v_4\}$ is the unique γ_t-set of G, so that $f_{\gamma_t}(P_n) = 0$.

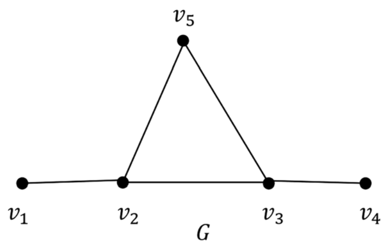


Fig. 1 A graph with $f_{\gamma_t}(G) = 0$ and $f_\gamma(G) = 1$

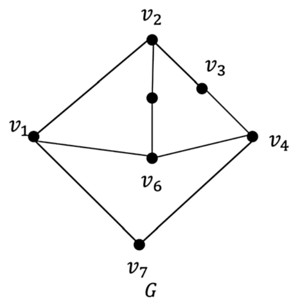


Fig. 2 A graph with $f_{\gamma t}(G) = 1$ and $f_\gamma(G) = 0$

Subcase 1 (ii). Let $n \neq 5$ and $n = 2m + 1$ and $m \geq 3$. Then $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, ..., v_{2m-1}, v_{2m}\}$ is the unique γ_t-set of G containing v_1 so that $f_{\gamma_t}(P_n) = 1$.

Case 2. n is even.

Subcase 2 (i). Let $n \equiv 0 \pmod 4$)

Let $n = 4m$ and $m \geq 1$. Then $S_1 = \{v_2, v_3, v_6, v_7, v_{10}, v_{11}, ..., v_{4m-2}, v_{4m-1}\}$ is the unique γ_t-set of G so that $f_{\gamma_t}(P_n) = 0$.

Subcase 2 (ii). Let $n \equiv 2 \pmod 4$.

Let $n = 4m + 2$ and $m \geq 2$. Let S be any γ_t-set of G. Then it is easily verified that any singleton subset of S is a subset of another γ_t-set of G and so $f_{\gamma_t}(P_n) \geq 1$. Then $S_1 = \{v_1, v_2, v_5, v_6, v_8, v_9, ..., v_{4m}, v_{4m+1}\}$ is a γ_t-set of G. Since S_1 is the unique γ_t-set of G containing $\{v_1, v_{4m+1}\}$, $f_{\gamma_t}(P_n) = 2$.

Let $n = 4m + 2$ and $m = 1$. Now $S_1 = \{v_1, v_2, v_5, v_6\}$, $S_2 = \{v_1, v_2, v_4, v_5\}$, $S_3 = \{v_2, v_3, v_4, v_5\}$ and $S_4 = \{v_2, v_3, v_5, v_6\}$ are the only four γ_t-sets of G such that $f_{\gamma_t}(S_1) = 2$,$f_{\gamma_t}(S_2) = 2$, $f_{\gamma_t}(S_3) = 2$, $f_{\gamma_t}(S_4) = 2$ so that $f_{\gamma_t}(P_n) = 2$. □

Theorem 4 *For the complete graph $G = K_n$ $(n \geq 2)$, $f_{\gamma_t}(G) = 1$.*

Proof Let $V(K_n) = \{v_1, v_2, ..., v_n\}$. Then $S_i = \{v_i\}(1 \leq i \leq n)$ is a γ_t-set of G and so $\gamma_t(G) = 1$. Since $n \geq 2$, G has at least two γ_t-sets. Then by Theorem 2, $f_{\gamma_t}(G) = 1$. □

Theorem 5 *For the complete bipartite graph $G = K_{r,s}(1 \leq r \leq s)$,*

$$f_{\gamma_t}(G) = \begin{cases} 0 & for \quad r = 1; s \geq 2 \\ 2 & for \quad 1 < r \leq s \end{cases}$$

Proof Let $U = \{u_1, u_2, ..., u_r\}$ and $V = \{v_1, v_2, ..., v_s\}$ be the bipartite sets of G.

Case1. Let $r = 1$ and $s \geq 2$. Then $S = \{u_1\}$ is the γ_t-set of G. So that $\gamma_t(G) = 0$.

Case 2. Let $1 < r \leq s$. Then $S_{ij} = \{u_i, v_j\}$ $(1 < i < r, 1 < j < s)$ is a γ_t-set of G and so $\gamma_t(G) = 2$. Since any singleton subset of S_{ij} is not a forcing subset of S_{ij},$f_{\gamma_t}(G) \geq 2$. Then by Theorem 2, $f_{\gamma_t}(G) = 2$. □

Theorem 6 *For the cycle $G = C_n(n \geq 4)$,*

$$f_{\gamma_t}(C_n) = \begin{cases} 2 & \text{if } n \text{ is odd or } n \equiv 0 \pmod 4 \\ 4 & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

Proof Let $V(C_n) = \{v_1, v_2, v_3, ..., v_n, v_1\}$.

Case 1. n is odd.

Let $n = 2m + 1, m \geq 2$. Then any singleton subset of S is a subset of another γ_t-set of G and so $f_{\gamma_t}(C_n) \geq 1$.

Subcase 1 (i). $n + 1 \equiv 0$ (mod 4).

Let $n = 4k - 1$, $k \geq 1$. Then $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, ..., v_{4k-3}, v_{4k-2}\}$ is the unique γ_t-set of G containing $\{v_1, v_{10}\}$ so that $f_{\gamma_t}(C_n) = 2$.

Subcase 1 (ii). $n - 1 \equiv 0$ (mod 4).

Let $n = 4k + 1$, $k \geq 1$. Then $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, ..., v_{4k-3}, v_{4k-2}, v_{4k+1}\}$ is the unique γ_t-set of G containing $\{v_2, v_{4k+1}\}$ so that $f_{\gamma_t}(C_n) = 2$.

Case 2. n is even.

Subcase 2(i). Let $n \equiv 0$ (mod 4).

Let $n = 4k, k \geq 1$. Then $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, ..., v_{4k-3}, v_{4k-2}\}$ is the unique minimum total dominating set of G containing $\{v_1, v_2\}$, $\{v_5, v_6\}$,...,$\{v_{4k-3}, v_{4k-2}\}$ so that $f_{\gamma_t}(C_n) = 2$.

Subcase 2 (ii). Let $n \equiv 2$ (mod 4).

Let $n = 4k + 2$, $k \geq 1$. Let S be any γ_t-set of G. Then any one element or two element or three element subset of S is a subset of another γ_t-set of G and so $f_{\gamma_t}(C_n) \geq 4$. Now $S_1 = \{v_1, v_2, v_5, v_6, v_9, v_{10}, ..., v_{4k+1}, v_{4k+2}\}$ is a γ_t-set of G. It is easily seen that S_1 is the unique γ_t-set of G containing $\{v_1, v_2, v_{4k+1}, v_{4k+2}\}$ so that $f_{\gamma_t}(C_n) = 4$. □

3 Some Results on the Forcing Total Domination Number of a Graph

Definition 2 A vertex $v \in G$ is said to be a *total dominating vertex* of G if v belongs to every γ_t-set of G.

Example 2 For the graph G given in Fig. 3, $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_2, v_4\}$ are the only two γ_t-sets of G. Since v_2 belongs to every γ_t-set of G, v_2 is the unique total dominating vertex of G.

The proofs of the following theorems are straight forward so we omit the proofs.

Theorem 7 *Let G be a connected graph. Then*

(i) $f_{\gamma_t}(G) = 0$ if and only if G has a unique γ_t-set.

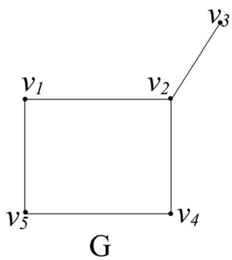


Fig. 3 A graph with total dominating vertex v_2

(ii) $f_{\gamma_t}(G) = 1$ if and only if G has at least two γ_t-sets, one of which is a unique γ_t-set containing one of its elements, and
(iii) $f_{\gamma_t}(G) = \gamma_t(G)$ if and only if no γ_t-set of G is the unique γ_t-set containing any of its proper subsets.

Theorem 8 *Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective minimum total dominating sets in G. Then $\cap_{F \in \Im} F$ is the set of total dominating vertices of G.*

Theorem 9 *Let G be a connected graph and W be the set of all total dominating vertices of G. Then $f_{\gamma_t}(G) \leq \gamma_t(G) - |W|$.*

Theorem 10 *Let G be a connected graph with at least one universal vertex. Then*

(i) $f_{\gamma_t}(G) = 0$ if and only if G contains exactly one universal vertex.
(ii) $f_{\gamma_t}(G) = 1$ if and only if G contains at least two universal vertices.

Theorem 11 *Let G be a connected graph with $\gamma_t(G) = 2$ and $c(G) = 4$, where $c(G)$ is the length of a smallest cycle in G.*

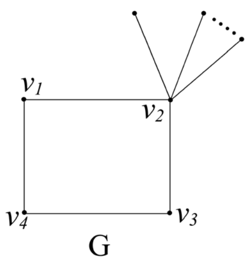
(i) Let $\Delta(G) = 2$. Then $f_{\gamma_t}(G) = 2$ if and only if $G = C_4$.
(ii) Let $\delta(G) = 1$. Then $f_{\gamma_t}(G) = 1$ if and only if G is the graph given Fig. 4.

Proof Let $C : u, v, w, z, u$ be a smallest cycle in G. Since $\gamma_t(G) = 2$, C is the only smallest cycle in G.

(i) Let $G = C_4$. Then by Theorem 6, $f_{\gamma_t}(G) = 2$. Conversely, let $f_{\gamma_t}(G) = 2$. Let S be a γ_t-set of G. Since $\gamma_t(G) = 2$, $G[S]$ is connected. Also since $\gamma_t(G) = f_{\gamma_t}(G) = 2$, by Theorem 7, no γ_t-set containing any of its proper subsets. Since $\gamma_t(G) = 2$ and $\Delta(G) = 2$, $S_1 = \{u, v\}$, $S_2 = \{v, w\}$, $S_3 = \{w, x\}$ and $S_4 = \{x, u\}$ are the only four γ_t-sets of G. Also since $c(G) = 4, uw, vx \notin E(G)$. If any one of the vertices u, v, w, x is a cut vertex of G, then any two S_i $(1 \leq i \leq 4)$ are the only γ_t-sets of G so that $f_{\gamma_t}(G) = 1$, which is a contradiction. Therefore G has no cut vertices. Hence it follows that $G = C_4$.

(ii) Let G be the graph given in the Fig. 4. Then $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_2, v_3\}$ are the only two γ_t-sets of G so that $f_{\gamma_t}(G) = 1$. Conversely, let $f_{\gamma_t}(G) = 1$. Let S be a

Fig. 4 A graph with $\delta(G) = 1$ and $f_{\gamma_t}(G) = 0$



γ_t-set of G. Since $f_{\gamma_t}(G) = 1$, by Theorem 7 (ii), G has at least two γ_t-sets such that one of which is the unique γ_t-set containing one of its elements. Since $\gamma_t(G) = 2$ and $G[S]$ is connected, $S_1 = \{u, v\}$ and $S_2 = \{v, w\}$ are the only two γ_t-sets of G. Since $\gamma_t(G) = 2$, G has no universal vertices. Also since $\delta(G) = 1$, there exists $x \in V$ such that x is adjacent to only the vertex v. Let us assume that $deg(x) = 1$. Therefore v is a cut vertex of G. Suppose that there exists $y \neq x$ such that $vy \in E(G)$. Since $c(G) = 4, yu, yw, yz, xy \notin E(G)$. Therefore G is the graph given in Fig. 4, which satisfies the requirements of this theorem. □

4 Realization Results

In this section, we present some graphs from which various graphs arising in later theorem are generated using identification.

Definition 3 Let $P_i : u_i, v_i, w_i$ $(1 \leq i \leq a)$ be a copy of path on three vertices. Let U_a be the graph obtained from P_i $(1 \leq i \leq a)$ by adding a new vertex x and join x with each u_i and w_i $(1 \leq i \leq a)$.

Definition 4 Let $Q_i : x_i, y_i, z_i, x_i$ $(1 \leq i \leq b)$ be a copy of cycle with three vertices. Let V_i $(1 \leq i \leq b)$ be a graph from Q_i $(1 \leq i \leq b)$ by introducing new vertices h_i and q_i and introducing the edges x_iz_i,x_ih_i, y_ih_i and z_iq_i. $(1 \leq i \leq b)$.

Definition 5 Let $C_i : p_i, q_i, r_i, s_i, t_i$,$f_i$,$p_i$ $(1 \leq i \leq c)$ be a copy of cycle with six vertices. Let Z_c be a graph obtained from C_i $(1 \leq i \leq c)$ by identifying s_{i-1} of C_{i-1} with p_i of C_i $(2 \leq i \leq c)$.

In view of Theorem 2, we have the following realization result.

Theorem 12 *For every pair a,b of integers with* $0 \leq a < b$ *and* $b \geq 1$, *there exists a connected graph G such that* $f_{\gamma_t}(G) = a$ *and* $\gamma_t(G) = b$.

Proof **Case 1.** $a = 0, b \geq 1$.

Subcase 1(i). $a = 0, b = 1$. Then the path $G = P_3$ satisfies the requirements of this theorem.

Subcase 1(ii). $a = 0, b = 2$. Let the cycle $C_5 = \{v_1, v_2, v_3, v_4, v_5, v_1\}$. Let G be the graph obtained from C_5 by introducing the edge v_1v_4. Then $S = \{v_1, v_4\}$ is the unique γ_t-set of G so that $f_{\gamma_t}(G) = 0$ and $\gamma_t(G) = 2$.

Subcase 1(iii). $a = 0, b \geq 3$. Let G be the graph obtained from $K_{1,b-1}$ by subdividing each edge. Then $V(K_{1,b-1})$ is the unique γ_t-set of G so that $f_{\gamma_t}(G) = 0$ and $\gamma_t(G) = b$.

Case 2. $0 < a < b$.

Subcase 2(i). $b = a + 1$. Consider the graph $G = U_a$. Let $H_i = \{u_i, w_i\}$ $(1 \leq i \leq a)$. Then every total dominating set of G contains the vertex x and also contains at least one element from each H_i $(1 \leq i \leq a)$ and so $\gamma_t(G) \geq a + 1$. Let $S = \{x\} \cup \{u_1, u_2, \ldots, u_a\}$. Then S is a total dominating set of G so that $\gamma_t(G) = a + 1$. Next we show that $f_{\gamma_t}(G) = a$. By Theorem 9, $f_{\gamma_t}(G) \leq \gamma_t(G) - |x| = a + 1 - 1 = a$. Now since $\gamma_t(G) = a + 1$ and every total dominating set of G contains x, it is easily seen that every γ_t -set of G is of the form $S_1 = \{x\} \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S_1 with $|T| < a$. Then there exists some i such that $T \cap H_i = \phi$, which shows that $f_{\gamma_t}(G) = a$.

Subcase 2(ii) $b \neq a + 1$. Let $P_i^{'} : h_i, q_i$ $(1 \leq i \leq b - a - 1)$ be a copy path of order 2. Let G be the graph obtained from U_a and $P_i^{'}$ $(1 \leq i \leq b - a - 1)$ by joining the vertex x with each h_i $(1 \leq i \leq b - a - 1)$. First we claim that $\gamma_t(G) = b$. Let $H_i = \{u_i, w_i\}$ $(1 \leq i \leq a)$. Let $X = \{x, h_1, h_2, ..., h_{b-a-1}\}$. It is easily observed that X is a subset of every total dominating set of G and every total dominating set of G contains at least one element from each H_i $(1 \leq i \leq a)$ and so $\gamma_t(G) \geq b - a + a = b$. Let $S_2 = X \cup \{u_1, u_2, ..., u_a\}$. Then S_2 is a total dominating set of G so that $\gamma_t(G) = b$. Next we show that $f_{\gamma_t}(G) = a$. By Theorem 9, $f_{\gamma_t}(G) \leq \gamma_t(G) - |X| = b - (b - a) = a$. Now since $\gamma_t(G) = b$ and every total dominating set of G contains X, it is easily seen that every γ_t -set of G is of the form $S_3 = X \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S_3 with $|T| < a$. Then it is clear that there exists some i such that $T \cap H_i = \phi$, which shows that $f_{\gamma_t}(G) = a$. □

By Theorem 4, for the complete graph $G = K_n (n \geq 2)$, $f_{\gamma_t}(G) = \gamma_t(G) = 1$. By Theorem 5, for the complete bipartite graph $G = K_{r,s} (2 \leq r \leq s)$, $f_{\gamma_t}(G) = \gamma_t(G) = 2$. Also from Theorem 6, for the cycle $G = C_6$, $f_{\gamma_t}(G) = \gamma_t(G) = 4$. So, we leave the following as a open question.

Problem 1 For every integer $a \geq 1$, does there exist a connected graph G such that $f_{\gamma_t}(G) = \gamma_t(G) = a$?

For a connected graph G, we know that $\gamma(G) \leq \gamma_t(G)$. But from Example 1, we observed that there is no relationship between $f_\gamma(G)$ and $f_{\gamma_t}(G)$. In the following we give some realization results.

Theorem 13 *For every integer* $a \geq 0$, *there exists a connected graph* G *such that* $f_\gamma(G) = f_{\gamma_t}(G) = a$.

Proof **Case 1.** $a = 0, b = 0$. Let $C_6 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_1\}$. Let G be the graph obtained from C_6 by introducing the vertex v_7 and the edges v_1v_7 and v_1v_4. Then $S = \{v_1, v_4\}$ is the unique γ-set as well as the unique γ_t-set of G so that $f_{\gamma_t}(G) = f_\gamma(G) = 0$.

Case 2. $a \geq 1$.

Subcase 2(i). $a = 1$. Then the graph G given in Fig. 3 satisfies this requirements of this theorem.

Subcase 2(ii). $a \geq 2$. Let $G = U_a (a \geq 2)$. First we show that $\gamma(G) = a$. Let $H_i = \{u_i, w_i\} (1 \leq i \leq a)$. Then every dominating set of G contains the vertexx x and contains at least one element from each $H_i (1 \leq i \leq a)$ and so $\gamma(G) \geq a + 1$. Let $S_1 = \{x\} \cup \{u_1, u_2, ..., u_a\}$. Then S_1 is a dominating set of G so that $\gamma(G) = a + 1$. Next we show that $f_\gamma(G) = a$. By Theorem 1, $f_\gamma(G) \leq \gamma(G) - 1 = a + 1 - 1 = a$. Now since $\gamma(G) = a + 1$ and every dominating set of G contains x and contains at least one element from each H_i $(1 \leq i \leq a)$, it is easily seen that every γ-set of G is of the form $S_2 = \{x\} \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i$ $(1 \le i \le a)$. Let T be any proper subset of S_2 with $|T| < a$. Then it is clear that there exists some i such that $T \cap H_i = \phi$, which shows that $f_\gamma(G) = a$.

Next we claim that $\gamma_t(G) = a + 1$. Now every total dominating set of G contains the vertex x and contains at leat one element from each H_i $(1 \le i \le a)$ and so $\gamma_t(G) \ge a + 1$. Now $S_3 = \{x\} \cup \{u_1, u_2, ..., u_a\}$ is a total dominating set of G so that $\gamma_t(G) = a + 1$. Next we show that $f_{\gamma_t}(G) = a$. By Theorem 9, $f_{\gamma_t}(G) \le \gamma_t(G) - 1 = a + 1 - 1 = a$. Now since $\gamma_t(G) = a + 1$ and every total dominating set of G contains x and contains at least one element from each H_i $(1 \le i \le a)$, every γ_t-set of G is of the form $S_4 = \{x\} \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i$ $(1 \le i \le a)$. Let T be any proper subset of S_4 with $|T| < a$. Then there exists some i such that $T \cap H_i = \phi$, which shows that $f_{\gamma_t}(G) = a$. □

Theorem 14 *For every pair a,b of non negative integers, there exists a connected graph G such that* $f_{\gamma_t}(G) = a$ *and* $f_\gamma(G) = b$.

Proof **Case 1.** $0 \le a \le b$

Subcase 1(i). $0 \le a = b$. Then the graph G constructed in Theorem 13 satisfies the requirements of this theorem.

Subcase 1(ii). $a = 0, b = 1$. Then the graph G constructed in Subcase 1(i) of the Theorem 12 satisfies the requirement of this theorem.

Subcase 1(iii). $a = 0, b \ge 2$. Let G be the graph obtained from the copy V_i $(1 \le i \le b)$ by adding a new vertex x and introducing the edges xh_i, xy_i, xq_i $(1 \le i \le b)$. First we show that $\gamma(G) = b + 1$. Let $T_i = \{x_i, y_i, z_i\}$ $(1 \le i \le b)$. Then every dominating set of G contains the vertex x and contains at least one vertex from each T_i $(1 \le i \le b)$ and so $\gamma(G) \ge b + 1$. Let $S = \{x, y_1, y_2, ..., y_b\}$. Then S is a γ-set of G so that $\gamma(G) = b + 1$. Next we show that $f_\gamma(G) = b$. By Theorem 1, $f_\gamma(G) \le \gamma(G) - 1 = b + 1 - 1 = b$. Now since $\gamma(G) = b + 1$ and every dominating set of G contains x, it is easily seen that every γ-set of G is of the form $S_1 = \{x\} \cup \{c_1, c_2, ..., c_b\}$, where $c_i \in T_i$ $(1 \le i \le b)$. Let T be any proper subset of S_1 with $|T| < b$. Then it is clear that there exists some i such that $T \cap T_i = \phi$, which shows that $f_\gamma(G) = b$. Next we prove that $\gamma_t(G) = b + 1$ and $f_{\gamma_t}(G) = 0$. Now every γ_t-set of G contains the vertex x and contains only the vertex y_i from each T_i $(1 \le i \le b)$ and so $\gamma_t(a) \ge b + 1$. Then $S = \{x, y_1, y_2, ..y_b\}$ is the unique γ_t-set of G so that $\gamma_t(G) = b + 1$ and $f_{\gamma_t}(G) = 0$.

Subcase 1(iv). $0 < a < b$. Let G be the graph obtained from U_a and the copy V_i $(1 \le i \le b - a)$ by introducing the edges xh_i, xy_i and xq_i $(1 \le i \le b - a)$. Let $H_i = \{u_i, w_i\}$ $(1 \le i \le a)$ and $T_i = \{x_i, y_i, z_i\}$ $(1 \le i \le b - a)$. Then every dominating set of G contains the vertex x and at least one element from each $H_i (1 \le i \le a)$ and contains at least one element from each $T_i (1 \le i \le b - a)$ and so $\gamma(G) \ge b - a + a + 1 = b + 1$. Now $S_2 = \{x\} \cup \{u_1, u_2, ..., u_a\} \cup \{y_1, y_2, ..., y_{b-a}\}$ is a dominating set of G so that $\gamma(G) = b + 1$. Next we show that $f_\gamma(G) = b$. By Theorem 1, $f_\gamma(G) \le \gamma(G) - 1 = b + 1 - 1 = b$. Now since $\gamma(G) = b + 1$ and every dominating set of G contains x, every γ-set of G is of the form $S_3 = \{x\} \cup \{c_1, c_2, ..., c_a\} \cup \{d_1, d_2, ..., d_{b-a}\}$, where $c_i \in H_i$ $(1 \le i \le a)$ and $d_i \in T_i$ $(1 \le i \le b - a)$. Let T be any proper subset of S_3 with $|T| < b$. Then it is clear that there exist some i and j such that $T \cap H_i \cap T_j = \phi$, which shows that $f_{\gamma}(G) = b$.

Next we show that $\gamma_t(G) = b + 1$. Let $X = \{x, y_1, y_2, ..., y_{b-a}\}$. Then X is a subset of every total dominating set of G and every total dominating set of G contains at least one vertex from each H_i $(1 \leq i \leq a)$ and so $\gamma_t(G) \geq b - a + a + 1 = b + 1$. Let $S_4 = X \cup \{u_1, u_2, ..., u_a\}$. Then S_4 is a total dominating set of G so that $\gamma_t(G) = b + 1$. Next we show that $f_{\gamma_t}(G) = a$. By Theorem 9, $f_{\gamma_t}(G) \leq \gamma_t(G) - |X|$ $= b + 1 - (b - a + 1) = a$. Now since $\gamma_t(G) = b + 1$ and every total dominating set of G contains X, every γ_t-set of G is of the form $S_5 = X \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i$. Let T be any proper subset of G with $|T| < a$. Then there exists some i such that $T \cap H_i = \phi$, which shows that $f_{\gamma_t}(G) = a$.

Case 2. $0 \leq b \leq a$

Subcase 2(i). $0 \leq b = a$. Then the graph G constructed in Theorem 13 satisfies the requirements of this theorem.

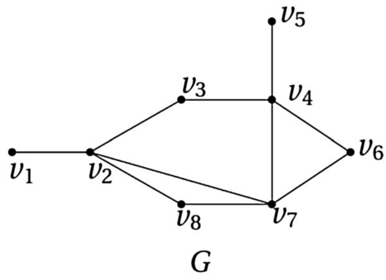
Subcase 2(ii). $b = 0, a = 1$. Then the graph G given in Fig. 2 satisfies the requirements of this theorem.

Subcase 2(iii). $b = 0, a \geq 2$. Let Q be a graph obtained from C_6 by introducing the edge v_1v_4. Let H be the graph obtained from Z_a and Q by identifying the vertex s_a of Z_a and v_1 of Q. Let G be a graph obtained from H by introducing the vertex u and v and the edges p_1u and p_1v. First we show that $\gamma(G) = a + 2$ and $f_{\gamma}(G) = 0$. Let $X = \{p_1, p_2, p_3, ..., p_a, v_1, v_4\}$. Then X is a subset of every dominating set of G and so $\gamma(G) \geq a + 2$. Since X is the unique γ-set of G, $\gamma(G) = b + 2$ and $f_{\gamma}(G) = 0$. Next we claim that $\gamma_t(G) = 2b + 2$. Let $J_i = \{f_i, q_i\}$ $(1 \leq i \leq a)$. Let $X = \{p_1, p_2, p_3, ..., p_b, v_1, v_5\}$. Then X is a susbet of every total dominating set of G. Also it is easily seen that every total dominating set of G contains at least one vertex from each J_i $(1 \leq i \leq b)$ and so $\gamma_t(G) \geq 2a + 2$. Let $S_6 = X \cup \{f_1, f_2, ..., f_a\}$. Then S_6 is a total dominating set of G so that $\gamma_t(G) = 2a + 2$. Next we claim that $f_{\gamma_t}(G) = a$. By Theorem 9, $f_{\gamma_t}(G) \leq \gamma_t(G) - |X| = 2a + 2 - (a + 2) = a$. Now since $\gamma_t(G) = 2a + 2$ and every total dominating set of G contains X, every γ_t-set of G of the S_7 is of the form $S_7 = W \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in J_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S_7 with $|T| < a$. Then there exists some i such that $T \cap J_i = \phi$, which shows that $f_{\gamma_t}(G) = a$.

Subcase 2(iv). $0 < b < a$ Let H be the graph obtained from Z_{a-b} and Q by identifying the vertex s_{a-b} of Z_{a-b} and v_1 of Q. Let G be the graph obtained from H and U_b by introducing new vertices y, u and v and the edges yx, p_1u ans p_1v. First we claim that $f_{\gamma}(G) = b$ and let $H_i = \{u_i, w_i\}$ $(1 \leq i \leq b)$. Let $X = \{x, p_1, p_2, ..., p_{a-b}, v_1, v_4\}$. Then any γ-set is of the form $S_8 = X \cup \{c_1, c_2, ..., c_b\}$, where $c_i \in H_i (1 \leq i \leq b)$. Then as in earlier cases it can be seen that $f_{\gamma}(G) = b$. Next we show that $f_{\gamma_t}(G) = a$. Let $J_i = \{f_i, q_i\}$ $(1 \leq i \leq a - b)$. Then any γ_t-set of G is of the form is of the form $W = X \cup \{c_1, c_2, ..., c_b\} \cup \{d_1, d_2, ..., d_{a-b}\}$, where $c_i \in H_i$ $(1 \leq i \leq b)$ and $d_i \in J_i$ $(1 \leq i \leq a - b)$. Then as in earlier cases it can be easily seen that $f_{\gamma_t}(G) = a$. □

Remark 1 For the graph G given in Fig. 5, $S = \{v_2, v_4\}$ is the unique γ-set of G so that $\gamma(G) = 2$ and $f_{\gamma}(G) = 0$. Also $S_1 = \{v_2, v_3, v_4\}$ and $S_2 = \{v_2, v_4, v_5\}$ are the only two γ_t-sets of G so that $\gamma_t(G) = 3$ and $f_{\gamma_t}(G) = 1$. Thus

Fig. 5 A graph with $f_{\gamma}(G) = 0$ and $f_{\gamma_t}(G) = 1$



$f_{\gamma}(G) < f_{\gamma_t}(G) < \gamma(G) < \gamma_t(G)$. For the path $G = P_7$, by Theorem 3, $\gamma_t(G) = 4$ and $f_{\gamma_t}(G) = 1$. Also $\gamma(G) = 3$ and $f_{\gamma}(G) = 2$. Thus $f_{\gamma_t}(G) < f_{\gamma}(G) < \gamma(G) < \gamma_t(G)$. For the cycle $G = C_6$, by the Theorem 6, $\gamma_t(G) = f_{\gamma_t}(G) = 4$. Also $\gamma(G) = 3, f_{\gamma}(G) = 1$, Thus $f_{\gamma}(G) < \gamma(G) < f_{\gamma_t}(G) = \gamma_t(G)$.

So we leave the following as an open question.

Problem 2 For every four positive integers a,b,c,d with $a \geq 0$, $b \geq 0$, $1 \leq c \leq d$, $0 \leq b \leq d$ and $d \geq 1$, does there exist a connected graph G such that $f_{\gamma}(G) = a$ and $f_{\gamma_t}(G) = b$, $\gamma(G) = c$, $\gamma_t(G) = d$?

5 The Upper Forcing Total Domination Number of a Graph

In [8], Zhang introduced the concept of the upper forcing geodetic number of a graph. In a similar manner we define the upper forcing total domination number of a graph as follows.

Definition 6 Let G be a connected graph with at least two vertices and S a γ_t-set of G. A *forcing subset* $T \subseteq S$ uniquely determines S containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing total domination number* of S, denoted by $f_{\gamma_t}(S)$, is the cardinality of a minimum forcing subset of S. The *forcing total domination number* of G, denoted by $f_{\gamma_t}(G)$ is defined by $f_{\gamma_t}(G)$ = min $\{f_{\gamma_t}(S)\}$, where the minimum is taken over all minimum total dominating sets S in G and *the upper forcing total domination number* of G, denoted by $f^{+}_{\gamma_t}(G) = max\{f_{\gamma_t}(S)\}$, where the maximum is taken over all γ_t-sets S in G.

The next theorem follows immediately from the definition of the total domination, forcing total domination and the upper forcing total domination numbers of a graph G.

Theorem 15 *For every connected graph* G, $0 \leq f_{\gamma_t}(G) \leq f^{+}_{\gamma_t}(G) \leq \gamma_t(G)$ *and* $\gamma_t(G) = 3$

Remark 2 For the graph G given Fig. 2, $f_{\gamma_t}(G) = 1, f^{+}_{\gamma_t}(G) = 2$ and $\gamma_t(G) = 3$. So, we leave the following as an open question.

Problem 3 For any three positive integers a,b and c with $0 \leq a \leq b \leq c$ and $c \geq 1$ does, there exists a connected graph G with $f_{\gamma_t}(G) = a$, $f^+_{\gamma_t}(G) = b$ and $\gamma_t(G) = c$?

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References

1. Chartrand, G., Galvas, H., Harary, F., Vandell, R.C.: The forcing domination number of a graph. J. Comb. Math. Comb. Comput. **25**, 161–174 (1997)
2. Cochaye, E.J., Dawes, R.M., Hedetnieni, S.T.: Total domination in graphs. Networks **10**, 211–219 (1980)
3. Favaron, O., Henning, M.A., Mynhart, C., Puech, J.: Total domination in graphs with minimum degree three. J. Graph Theory **34**, 9–19 (2000)
4. Haynes, T.W., Hedetniemi, S.T., Slater, P.J.: Fundamentals of Domination in Graphs. Marcel Dekker, New York (1998)
5. Henning, M.A., Yeo, A.: A new upper bound on the total domination number of a graph. Electron. J. Comb. **14**, 1–10 (2007)
6. Henning, M.A.: A survey of selected recent results on total domination in graphs. Discret. Math. **309**, 32–63 (2009)
7. Lam, P.C.B., Wei, B.: On the total domination number of graphs. Utilitas Mathematica **72**, 223–240 (2007)
8. Zhang, P.: The upper forcing geodetic number of a graph. Ars Comb. **62**, 3–15 (2012)